Algorithms for Designing Wavelets to Match a Specified Signal

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***Abstract*—Algorithms for designing a mother wavelet $\psi(x)$ such that it matches a signal of interest and such that the family of wavelets $\{2^{-(j/2)}\psi(2^{-j}x-k)\}$ forms an orthonormal Riesz basis of $L^2(\Re)$ are developed. The algorithms are based on a closed form solution for finding the scaling function spectrum from the wavelet spectrum. Many applications of signal representation, adaptive coding and pattern recognition require wavelets that are matched to a signal of interest. Most current design techniques, however, do not design the wavelet directly. They either build a composite wavelet from a library of previously designed wavelets, modify the bases in an existing multiresolution analysis or design a scaling function that generates a multiresolution analysis with some desired properties. In this paper, two sets of equations are developed that allow us to design the wavelet directly from the signal of interest. Both sets impose bandlimitedness, resulting in closed form solutions. The first set derives expressions for continuous matched wavelet spectrum amplitudes. The second set of equations provides a direct discrete algorithm for calculating close approximations to the optimal complex wavelet spectrum. The discrete solution for the matched wavelet spectrum amplitude is identical to that of the continuous solution at the sampled frequencies. An interesting byproduct of this work is the result that Meyer's spectrum amplitude construction for an orthonormal bandlimited wavelet is not only sufficient but necessary. Specific examples are given which demonstrate the performance of the wavelet matching algorithms for both known orthonormal wavelets and arbitrary signals.**

***Index Terms*—Bandlimited wavelets, constrained optimization, matched wavelets, orthonormal wavelets.**

I. Introduction

IN [1, Ch. 1] Daubechies introduces the wavelet transform as "a tool that cuts up data or functions or operators into different frequency components, and then studies each component with a resolution matched to its scale" [1]. One of the exciting advantages of wavelets over Fourier analysis is the flexibility they afford in the shape and form of the analyzer, that which "cuts up" and "studies" the signal of interest. However, with flexibility comes the difficult task of choosing or designing the appropriate wavelet or wavelets for a given application.

If we look at a multiresolution decomposition as the output of a bank of matched filters [2], we can see the need for the analyzing wavelet to "look" like the signal of interest. In signal detection applications, the decomposition of a signal in the presence of noise using a wavelet matched to the signal would produce a sharper or taller peak in time-scale space as compared to standard nonmatched wavelets. The design techniques developed to date do not specifically address the need for maximizing correlation in a signal decomposition.

Daubechies' classic technique [3] for finding orthonormal wavelet bases with compact support is often used as the default in many wavelet applications. However, the wavelets produced are independent of the signal being analyzed. Tewfik, Sinha, and Jorgensen [4] have developed a technique for finding the optimal orthonormal wavelet basis for representing a specified signal within a finite number of scales. Gopinath, Odegard, and Burrus [5] extended the results of Tewfik, *et al.*, by assuming bandlimited signals and finding the optimal M-band wavelet basis for representing a desired signal, again within a finite number of scales. Both of these approaches seek to represent a signal over some number of scales. However, the desired output of a multiresolution decomposition of a bandpass signal using a matched wavelet is representation in one or at most two scales.

The wavelet design techniques developed Mallat and Zheng [6], and Chen and Donoho [7], build nonorthonormal wavelet bases from a library of existing wavelets in such a way that some error cost function is minimized. These techniques are constrained by the library of functions used and do not satisfy the need for optimal correlation in both scale and translation. Sweldens developed the lifting scheme for constructing biorthogonal wavelets [8]. Aldroubi and Unser [9] match a wavelet basis to a desired signal by either projecting the desired signal onto an existing wavelet basis, or transforming the wavelet basis under certain conditions such that the error norm between the desired signal and the new wavelet basis is minimum. Both of these techniques are constrained by their initial choice of MRA.

Apart from being of mathematical interest, the problem of deriving orthonormal wavelets directly from a signal of interest has specific application to signal detection, image enhancement, and target detection, to name a few. In this paper, we will show that in the case of orthonormal MRA's with bandlimited wavelets,there is a solution to finding wavelets that "look" like a desired signal. The only additional constraints are the *necessary conditions* for an MRA and the signal of interest

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itself. While the matching algorithm is sub-optimal in the sense that it is performed on the spectrum magnitude and phase independent of one another, we will show by way of examples that it produces good matching wavelets.

While application of the discrete wavelet transform has so far been focused mostly on the use of wavelets that are compactly supported in time (a key reason being the associated digital filter bank that makes for easy implementation), there are several situations where bandlimited wavelets and scaling functions are better suited. Examples can be found in fields such as communication, signal analysis and pattern recognition [10]–[16]. Even from an implementation viewpoint, new schemes such as perfect reconstruction circular convolution filter banks [17], [18] offer a potentially attractive computational structure for bandlimited wavelets. Thus there are several motivating factors for further research into bandlimited wavelets and this paper investigates their construction. The principal result is a frequency domain construction that can be viewed as a generalization of the orthogonal Meyer wavelets.

This paper is organized into eight sections. Section II contains background information on orthonormal multiresolution analyzes for use throughout the paper. We develop the rationale for matching both the amplitude and phase of the wavelet spectrum in Section III by applying matched filter theory to the wavelet decomposition. In Section IV, we develop the properties of an orthonormal wavelet that form the foundation for our matching algorithms, which are developed in Section V for the continuous case and Section VI for the discrete case. Section VII gives specific examples demonstrating the performance of the matching algorithm, followed by a summary in Section VIII.

II. ORTHONORMAL MRAS

Wavelet transform theory and its application to multiresolution signal decomposition has been thoroughly developed and well documented over the past decade [1], [3], [19], [20]–[22]. We will summarize the results of that work for the dyadic case here for use later in the paper. In an orthonormal MRA (OMRA), a signal, $f(x) \in V_{-1}$, is decomposed into an infinite series of detail functions, $\{g_j(x)\}$, such that [1], [3], [19], [20]–[22]

$$f(x) = \sum_{j=0}^{\infty} g_j(x). \tag{1}$$

The first level decomposition is done by projecting $f(x)$ onto two orthogonal subspaces, V_0 and W_0, where $V_{-1} = V_0 \oplus W_0$ and $(\oplus)$ is the direct sum operator. The projection produces $f_0(x) \in V_0$, a low resolution approximation of $f(x)$, and $g_0(x) \in W_0$, the detail lost in going from $f(x)$ to $f_0(x)$. The decomposition continues by projecting $f_0(x)$ onto V_1 and W_1 and so on. The orthonormal bases of W_j and V_j are given by

$$\begin{aligned} \psi_{j,k} &= 2^{-j/2}\psi(2^{-j}x-k)\\ \phi_{j,k} &= 2^{-j/2}\phi(2^{-j}x-k) \end{aligned} \tag{2}$$

where $\psi(x)$ is the mother wavelet and $\phi(x)$ is the scaling function [1], [3], [19], [20]–[22], where

$$\begin{aligned} \int \psi(x)\,dx = 0 &\iff \Psi(0)=0\\ \int \phi(x)\,dx = 1 &\iff \Phi(0)=1 \end{aligned} \tag{3}$$

and $\Phi(\omega)$ and $\Psi(\omega)$ are the Fourier Transform of $\phi(x)$ and $\psi(x)$, respectively. The projection equations are

$$\begin{aligned} g_j(x) &= \sum_{k=-\infty}^{\infty} d_k^j 2^{-(j/2)}\psi(2^{-j}x-k)\\ d_k^j &= \langle f_{j-1}(x), \psi_{j,k}\rangle \end{aligned} \tag{4}$$

$$\begin{aligned} f_j(x) &= \sum_{k=-\infty}^{\infty} c_k^j 2^{-(j/2)}\phi(2^{-j}x-k)\\ c_k^j &= \langle f_{j-1}(x), \phi_{j,k}\rangle \end{aligned} \tag{5}$$

where d_k^j and c_k^j are the projection coefficients and $\langle\cdot,\cdot\rangle$ is the L^2 inner product. The nested sequence of subspaces, $\{V_j\}$, constitutes the multiresolution analysis. For the MRA to be orthonormal 1) $\psi_{j,k}$ and $\phi_{j,k}$ must be orthonormal bases of W_j and V_j, respectively, and 2) $W_j \perp W_k$, for $j \neq k$; and 3) $W_j \perp V_j$, which leads to the following conditions on ψ and ϕ [22], [23]

$$\langle \phi_{j,k}, \phi_{j,m}\rangle = \delta_{k,m} \tag{6}$$

$$\langle \phi_{j,k}, \psi_{j,m}\rangle = 0 \tag{7}$$

$$\langle \psi_{j,k}, \psi_{\ell,m}\rangle = \delta_{j,\ell}\cdot\delta_{k,m} \tag{8}$$

The Fourier transform of (6) gives the Poisson summation, which is 1 for all ω,

$$\sum_{m=-\infty}^{\infty} |\Phi(\omega+2\pi m)|^2 = 1. \tag{9}$$

Since $\phi(x) \in V_0 \subset V_{-1}$ and $\psi(x) \in W_0 \subset V_{-1}$, they can be represented as linear combinations of the basis of V_{-1}

$$\phi(x) = 2\sum_{k=-\infty}^{\infty} h_k\phi(2x-k) \tag{10}$$

$$\psi(x) = 2\sum_{k=-\infty}^{\infty} g_k\phi(2x-k). \tag{11}$$

In the frequency domain (10) and (11) become

$$\begin{aligned} \Phi(\omega) &= H\left(\frac{\omega}{2}\right)\Phi\left(\frac{\omega}{2}\right)\\ \Psi(\omega) &= G\left(\frac{\omega}{2}\right)\Phi\left(\frac{\omega}{2}\right). \end{aligned} \tag{12}$$

For orthonormal MRAs, the sequences h_k and g_k in (10) and (11) represent the impulse responses of quadrature mirror filters (QMF) and have the following properties [19], [24]:

$$|H(\omega)|^2 + |G(\omega)|^2 = 1 \tag{13}$$

$$H(\omega)\overline{H(\omega+\pi)} + G(\omega)\overline{G(\omega+\pi)} = 0 \tag{14}$$

where $H(\omega)$ and $G(\omega)$ are the Fourier transforms of h_k and g_k, respectively, and are therefore both 2π-periodic. In this paper, we will choose

$$g_k = (-1)^{k+1} h_{1-k} \Longleftrightarrow G(\omega) = e^{i\omega}\overline{H(\omega+\pi)} \tag{15}$$

thereby guaranteeing (14) is always satisfied.

III. Signal Detection

Using a matched filter bank interpretation of wavelet transforms [2], we propose to design a wavelet that "matches" the signal of interest such that the output of the matched filter bank is maximized. The projection equation for the detail functions, given in (4), is an inner product integral and can be rewritten in the frequency domain by way of Parseval's Identity [22]

$$d_k^j = \langle f(x), \psi_{j,k}\rangle = \langle F(\omega), \Psi_{j,k}(2^j\omega)\rangle \tag{16}$$

where $\Psi_{j,k}(2^j\omega) = 2^{j/2} e^{-i2^j\omega k}\Psi(2^j\omega)$, is the Fourier transform of $\psi_{j,k}(x)$. The energy of d_k^j at a particular scale, j_0, and translation, k_0, is given by its squared magnitude

$$|d_{k_0}^{j_0}|^2 = |\langle F(\omega), \Psi_{j_0,k_0}(2^{j_0}\omega)\rangle|^2. \tag{17}$$

Applying the Cauchy–Schwarz inequality to the right side of (17) gives

$$\begin{aligned}&\langle F(\omega), \Psi_{j_0,k_0}(2^{j_0}\omega)\rangle|^2\\ &\quad\le \langle F(\omega), F(\omega)\rangle\langle \Psi_{j_0,k_0}(2^{j_0}\omega), \Psi_{j_0,k_0}(2^{j_0}\omega)\rangle\end{aligned} \tag{18}$$

where the equality holds if and only if

$$F(\omega) = K\Psi_{j_0,k_0}(2^{j_0}\omega) \tag{19}$$

where both F and Ψ are complex spectra. Therefore, $|d_{k_0}^{j_0}|^2$ is maximized when the complex frequency spectrum of ψ_{j_0,k_0} is identical to that of $f(x)$. Therefore, we would like to develop a method for matching the complex spectrum of the wavelet to that of the desired signal while maintaining the conditions for an orthonormal MRA. However, because the conditions for orthonormality are on the spectrum amplitude (Poisson summation) only, our solution matches the spectrum amplitudes and group delays independently. While this approach is not ideal from an optimization standpoint, we will show that it still leads to good matching wavelets.

One other difficulty in matching the wavelet spectrum directly to that of the desired signal is the fact that the conditions for an orthonormal MRA are on the scaling function and its frequency spectrum, not the wavelet specifically. If we were to construct a wavelet that satisfied its conditions for an orthonormal basis, it would not necessarily lead to a scaling function that generates an orthonormal MRA [1], [20]. Therefore, we must propagate the conditions for an orthonormal MRA from the 2-scale sequence and scaling function to the wavelet, match the wavelet to the desired signal under those conditions, and then calculate the scaling function and 2-scale sequence always guaranteeing that the conditions for an orthonormal MRA are satisfied.

IV. Properties of a Wavelet in an OMRA

Most wavelet construction techniques first find a scaling function that satisfies (3), (9), and (10) and then calculates the wavelet using (11) and (15). In this paper, we will show a wavelet construction technique that matches the wavelet directly to a signal of interest. To do so, the necessary and sufficient conditions for an OMRA, which are imposed exclusively on the scaling function, must be transferred to the wavelet. These conditions will be derived for the wavelet spectrum amplitude in Section IV-B and the spectrum phase in Section IV-C.

A. Finding the Scaling Function from a Wavelet

The first step in deriving the OMRA conditions for the wavelet spectrum amplitude is providing a means of deriving the scaling function from the mother wavelet. Finding the wavelet from the scaling function is simple using (11), however, it is not invertible. To derive an expression for $|\Phi|$ in terms of $|\Psi|$, the conditions provided in Section II will be applied directly. Conditions (3), (6) and (12) are required for $\phi(x)$ to generate an orthonormal MRA, thereby satisfying (6)–(14), [23].

From (12) and (13), we get the following expression [1]:

$$|\Phi(\omega)|^2 = |\Psi(2\omega)|^2 + |\Phi(2\omega)|^2. \tag{20}$$

Repeated substitution of $|\Phi(2^k\omega)|^2$ for $k \ge 1$ into (20) gives the following closed form solution [1]

$$|\Phi(\omega)|^2 = \sum_{j=1}^{\infty} |\Psi(2^j\omega)|^2 \qquad \text{for } \omega \ne 0. \tag{21}$$

B. Properties of the Wavelet Spectrum Amplitude

Now that we have an expression for finding $|\Phi|$ from $|\Psi|$, we need to develop the constraints on $|\Psi|$ that are necessary and sufficient to guarantee $\phi_{j,k}$ is an orthonormal basis of V_j. Using (21), conditions (3), (6), and (12) can be transferred to conditions on $|\Psi(\omega)|$. To guarantee a closed form solution, we assume the scaling function spectrum is bandlimited with only a countable number of zeros. With this assumption, we can derive the following theorems for the properties of orthonormal bandlimited scaling function and wavelet spectra.

Theorem 1 (Bandlimited Φ): In a multiresolution analysis, the spectrum of a bandlimited scaling function, $\Phi(\omega)$, with at most a countable number of zeroes, has support on $\omega \in [-\omega_m, \omega_m]$ where

$$\omega_m \le \pi + \alpha \qquad 0 \le \alpha \le \frac{\pi}{3}. \tag{22}$$

Furthermore, the following conditions on $|\Phi(\omega)|$ are necessary and sufficient for a bandlimited orthonormal scaling function

$$\begin{aligned}|\Phi(\omega)| &= 1, &&\text{for } |\omega| < \pi - \alpha\\ |\Phi(\omega)|^2 + |\Phi(2\pi-\omega)|^2 &= 1, &&\text{for } \pi - \alpha < |\omega| < \pi + \alpha.\end{aligned} \tag{23}$$

Proof: Let the half-bandwidth of $\Phi(\omega)$ be ω_m. Because $H(\omega)$ is 2π-periodic, then the half-bandwidth of one of its periods is obviously π. From (12) we see that

$$\Phi(2\omega) = H(\omega)\Phi(\omega) \tag{24}$$

and therefore the half-bandwidth of $\Phi(2\omega)$ is $\omega/2 \le \pi$ for $\omega_m \le 2\pi$. The Poisson summation (9) can now be simplified for a bandlimited orthonormal scaling function as $|\Phi(\omega)|^2 + |\Phi(\omega + 2\pi)|^2 = 1$ for $0 \le \omega \le 2\pi$. Assuming there are only a countable number of zeros in the interval $|\omega| < \omega_m$, where ω_m is the half bandwidth of $|\Phi(\omega)|$, then clearly, $\pi \le \omega_m$.

So let the half bandwidth of $|\Phi(\omega)|$ be given as $\omega_m = \pi + \alpha$ where $\alpha \ge 0$. Then, from the simplified form of the Poisson summation given above

$$\begin{aligned} |\Phi(\omega)| &= 1, && \text{for } |\omega| \le \pi + \alpha \\ |\Phi(\omega)|^2 + |\Phi(2\pi - \omega)|^2 &= 1, && \text{for } \pi - \alpha \le \omega \le \pi + \alpha. \end{aligned} \tag{25}$$

Because $H(\omega)$ is the Fourier transform of a real, discrete sequence, it is 2π-periodic and symmetric, that is

$$H(\omega) = H(\omega + 2\pi) \quad H(\omega) = \overline{H(2\pi - \omega).} \tag{26}$$

Furthermore, from (12)

$$H(\omega) = \frac{\Phi(2\omega)}{\Phi(\omega)} \tag{27}$$

where again the half bandwidth of $\Phi(\omega)$ is given as $\omega_m = \pi + \alpha$, and so $\Phi(2\omega)$ is bandlimited such that $|\omega| \le (\pi + \alpha/2)$. Assuming $|\Phi(\omega)| = 0$ for a countable number of points on the interval $0 \le |\omega| \le \pi + \alpha$, then from (27), the following must be true:

$$\begin{aligned} H(\omega) &\ne 0 && \text{for } |\omega| \le \frac{\pi + \alpha}{2} \\ H(\omega) &= 0 && \text{for } \frac{\pi + \alpha}{2} \le |\omega| \le \pi + \alpha. \end{aligned} \tag{28}$$

By the symmetry requirement in (26b)

$$H\left(\frac{\pi + \alpha}{2}\right) = \overline{H\left(\frac{3\pi - \alpha}{2}\right)} = 0 \tag{29}$$

and so the only way for both (28) and (29) to be true is if $\alpha \le \pi/3$. Therefore, the necessary conditions on the half bandwidth of a bandlimited orthonormal scaling function is $\omega_m \le \pi + \alpha$, for $0 \le \alpha \le (\pi/3)$. □

Theorem 1 states that a bandlimited orthonormal scaling function spectrum has support that can vary from $[-\pi, \pi]$ to $[-4\pi/3, 4\pi/3]$ with the structure given in (23). Because the orthonormality conditions are in terms of the magnitude of Φ only and since Φ is bandlimited, the phase of Φ can take on any function. However, in the next section we will show that the scaling function phase can be derived directly from the wavelet phase.

It should be no surprise that a bandlimited scaling function that generates an orthonormal MRA also gives rise to a bandlimited wavelet.

Corollary 2 (Bandlimited Ψ): In an orthonormal MRA with a bandlimited scaling function, the corresponding orthonormal wavelet has support $|\omega| \in [\pi - \alpha, 2\pi + 2\alpha]$ where

$$0 \le \alpha \le \frac{\pi}{3} \tag{30}$$

and α in (30) takes on the same value as α in (22). Furthermore, $|\Psi(\omega)|$ can be expressed in terms of $|\Phi|$ as

$$|\Psi(\omega)| = \begin{cases} 0, & 0 \le |\omega| < \pi - \alpha \\ |\Phi(2\pi - \omega)|, & \pi - \alpha \le |\omega| < \pi + \alpha \\ 1, & \pi + \alpha \le |\omega| \le 2\pi - 2\alpha \\ \left|\Phi\left(\frac{\omega}{2}\right)\right|, & 2\pi - 2\alpha \le |\omega| < 2\pi + 2\alpha \\ 0, & 2\pi + 2\alpha \le |\omega|. \end{cases} \tag{31}$$

The construction of $|\Psi(\omega)|$ given in (31) exhibits a symmetry about $4\pi/3$. This symmetry leads to the following necessary and sufficient conditions on $|\Psi|$:

$$\begin{aligned} |\Psi(\omega)| &= |\Psi(4\pi - 2\omega)| && \text{for } \pi - \alpha < |\omega| < \frac{4\pi}{3} \\ |\Psi(\omega)|^2 + |\Psi(2\pi - \omega)|^2 &= 1 && \text{for } \pi - \alpha < |\omega| < \frac{4\pi}{3}. \end{aligned} \tag{32}$$

Proof: The proof is easily derived by applying the conditions on $|\Phi(\omega)|^2$ from Theorem 1. The full proof can be found in [14]. □

Corollary 2 says that the spectrum amplitude of an orthonormal wavelet defined in the context of an OMRA has support on $|\omega|$ that can vary from $|\omega| \in [\pi, 2\pi]$ as the lower limit to $|\omega| \in [2\pi/3, 8\pi/3]$ as the upper limit.

Theorem 1 and Corollary 2 completely characterize the spectrum amplitudes of bandlimited scaling functions and wavelets in an orthonormal MRA and provide the necessary and sufficient conditions for their construction.

Two well known wavelets can be derived from Theorem 1 and Corollary 2 by setting α to its minimum and maximum values, respectively. The first is Shannon's wavelet, found by setting $\alpha = 0$. The second example, shown in detail here, can be constructed by setting $\alpha = \pi/3$. The scaling function spectrum amplitude can be written as

$$|\Phi(\omega)| = \begin{cases} 1 & 0 \le |\omega| < \frac{2\pi}{3} \\ \gamma(\omega) & \frac{2\pi}{3} < |\omega| \le \frac{4\pi}{3} \end{cases} \tag{33}$$

where $\gamma(\omega)$ represents the skirt of the scaling function amplitude spectrum and therefore

$$0 \le \gamma(\omega) \le 1 \quad \frac{2\pi}{3} \le |\omega| \le \frac{4\pi}{3}. \tag{34}$$

From (23)

$$\gamma(\omega)^2 + \gamma(2\pi - \omega)^2 = 1. \tag{35}$$

From (31), the wavelet spectrum amplitude can be written as

$$|\Psi(\omega)| = \begin{cases} 0, & 0 \leq |\omega| < \frac{2\pi}{3} \\ \gamma(2\pi - \omega), & \frac{2\pi}{3} \leq |\omega| < \frac{4\pi}{3} \\ \gamma\left(\frac{\omega}{2}\right), & \frac{4\pi}{3} < |\omega| \leq \frac{8\pi}{3 =} \\ 0, & |\omega| \leq \frac{8\pi}{3} \end{cases} \tag{36}$$

and from (32) the necessary and sufficient conditions for the wavelet amplitude spectrum with support $|\omega| \in [2\pi/3, 8\pi/3]$ are given by

$$\begin{array}{l} |\Psi(\omega)| = |\Psi(4\pi - 2\omega)| \\ |\Psi(\omega)|^2 + |\Psi(2\pi - \omega)|^2 = 1 \end{array} \quad \frac{2\pi}{3} \leq |\omega| \leq \frac{4\pi}{3}. \tag{37}$$

The wavelet construction given in (36) and (37) can be shown to be Meyer's spectrum amplitude construction exactly. Therefore, while Meyer's spectrum amplitude construction has been postulated as *one* possible construction, we have shown here that *any* bandlimited wavelet that forms a Riesz basis must be a generalized Meyer's wavelet.

In the remainder of this paper, we will assume $\alpha = \pi/3$, so that the bandlimited wavelet spectrum has support $|\omega| \in [2\pi/3, 8\pi/3]$. The conditions given in (37) will be used in Section V to match the wavelet spectrum to a desired signal spectrum.

C. Properties of the Wavelet Spectrum Phase

It would be convenient if we could simply set the phase of Ψ to the phase of the desired signal spectrum, F, thereby cancelling the complex component of (19). However, just as in the previous section we showed that Ψ has specific constraints on its amplitude, here we will show that Ψ has specific constraints on the structure of its phase as well. First we will develop an expression for the group delay of $\Psi(\omega)$ in terms of the group delay of the scaling function spectrum, $\Phi(\omega)$. Substituting (15) into (12) gives

$$\Psi(2\omega) = e^{-i\omega} \frac{\overline{\Phi(2\omega + 2\pi)}}{\overline{\Phi(\omega + \pi)}} \Phi(\omega) \tag{38}$$

and so the phase of Ψ becomes

$$\begin{aligned} \theta_\Psi(\omega) = &-\frac{\omega}{2} - \theta_\Phi(\omega + 2\pi) \\ &+ \theta_\Phi\left(\frac{\omega}{2} + \pi\right) + \theta_\Phi\left(\frac{\omega}{2}\right) \end{aligned} \tag{39}$$

where $\theta_\Psi(\omega)$ and $\theta_\Phi(\omega)$ are the phases of Ψ and Φ, respectively. The negatives of the group delays are denoted as Λ_Ψ and Λ_Φ. Setting $\Gamma_\Psi(\omega) = \Lambda_\Psi(\omega) + 1/2$ gives

$$\Gamma_\Psi(\omega) = -\Lambda_\Phi(\omega + 2\pi) + \frac{1}{2}\Lambda_\Phi\left(\frac{\omega}{2} + \pi\right) + \frac{1}{2}\Lambda_\Phi\left(\frac{\omega}{2}\right). \tag{40}$$

Next, we will develop an expression for the group delay of $\Psi(\omega)$ in terms of the group delay of $H(\omega)$, denoted as $\lambda(\omega)$. By repeated substitutions of the equations in (12) and (15), we get the following infinite products [3], [22]

$$\Phi(\omega) = \prod_{m=1}^{\infty} H\left(\frac{\omega}{2^m}\right) \tag{41}$$

$$\Psi(\omega) = e^{-i(\omega/2)} \overline{H\left(\frac{\omega}{2} + \pi\right)} \prod_{m=2}^{\infty} H\left(\frac{\omega}{2^m}\right) \tag{42}$$

where $H(\omega)$ is 2π-periodic. Taking the derivative of the phase of both sides gives the group delay

$$\Lambda_\Phi(\omega) = \sum_{m=1}^{\infty} 2^{-m} \lambda\left(\frac{\omega}{2^m}\right) \tag{43}$$

$$\Lambda_\Psi(\omega) = -\frac{1}{2} - \frac{1}{2}\lambda\left(\frac{\omega}{2} + \pi\right) + \sum_{m=2}^{\infty} 2^{-m} \lambda\left(\frac{\omega}{2^m}\right) \tag{44}$$

$$\Gamma_\Psi(\omega) = -\frac{1}{2}\lambda\left(\frac{\omega}{2} + \pi\right) + \sum_{m=2}^{\infty} 2^{-m} \lambda\left(\frac{\omega}{2^m}\right) \tag{45}$$

where $\Lambda_\Phi(\omega) = d\theta_\Phi(\omega)/d\omega$ $\Lambda_\Psi = d\theta_\Psi(\omega)/d\omega$, $\Gamma_\Psi(\omega) = \Lambda_\Psi + 1/2$ and $\lambda(\omega) = d\theta_H(\omega)/d\omega$ is 2π-periodic. We will match the group delay of the desired signal with $\Gamma_\Psi(\omega)$. Before proceeding, it is important to note some of the properties of the group delays of Φ, Ψ and H.

Theorem 3 (Properties of Λ_Φ, Λ_Ψ and λ): Let $\Lambda_\Phi(\omega) = d\theta_\Phi(\omega)/d\omega$ and $\Lambda_\Psi(\omega) = d\theta_\Psi(\omega)/d\omega$, where $\theta_\Phi(\omega)$ and $\theta_\Psi(\omega)$ are the phase functions of $\Phi(\omega)$and $\Psi(\omega)$, respectively. Let $\lambda(\omega) = d\theta_H(\omega)/d\omega$ where $\theta_H(\omega)$ is the 2π-periodic phase of $H(\omega)$. Then $\Lambda_\Phi(\omega)$, $\Lambda_\Psi(\omega)$ and $\lambda(\omega)$ have the following properties:

$$\begin{gathered} \Lambda_\Phi(\omega) = \Lambda_\Phi(-\omega), \quad \Lambda_\Psi(\omega) = \Lambda_\Psi(-\omega) \\ \lambda(\omega) = \lambda(-\omega) \end{gathered} \tag{46}$$

$$\int_{-\infty}^{\infty} \Lambda_\Phi(\omega)\, d\omega = \frac{1}{2\pi} \int_{-\pi}^{\pi} \lambda(\omega)\, d\omega = \overline{\lambda} \quad \text{where } \overline{\lambda} \in \mathbb{Z} \tag{47}$$

$$\int_{-\infty}^{\infty} \Lambda_\Psi(\omega)\, d\omega = -\tfrac{1}{2} \tag{48}$$

and therefore

$$\int_{-\infty}^{\infty} \Gamma_\Psi(\omega) d\omega = 0. \tag{49}$$

Proof: The proofs for the equations in (46) are trivial. Since $\phi(x)$, $\psi(x)$, and h_k are all real quantities, it can be shown that $\Lambda_\Phi(\omega)$, $\Lambda_\Psi(\omega)$, and $\lambda(\omega)$ are all even functions [30] and equations (47) and (48) can be proven by substituting $\lambda_0(\omega) = \lambda(\omega) - \overline{\lambda}$ into (43)–(45) and integrating. The full proof of Theorem 3 can be found in [14]. □

V. Matching Wavelets

The conditions for wavelet construction developed in the previous sections can now be applied to matching an orthonormal bandlimited wavelet to a desired signal, $f(x)$.

A. Matching Spectrum Amplitudes

To find the wavelet spectrum amplitude that best matches that of the desired signal, we minimize the following error function:

$$E(Y, a) = \int_{2\pi/3}^{8\pi/3} [W(\omega) - aY(\omega)]^2 \, d\omega \tag{50}$$

where $W(\omega) = |F(\omega)|^2$, $Y(\omega) = |\Psi(\omega)|^2$, and a is the scale factor. Because we are assuming real wavelets only, the wavelet spectrum is symmetric and the integral need only cover positive frequencies. The error function given in (50) can be written in a piecewise fashion as

$$\begin{aligned} E(Y, a) = \int_{2\pi/3}^{\pi} & [W(\omega) - aY(\omega)]^2 \\ & + [W(2\pi - \omega) - aY(2\pi - \omega)]^2 \\ & + [W(2\omega) - aY(2\omega)]^2 \\ & + [W(4\pi - 2\omega) - aY(4\pi - 2\omega)]^2 \, d\omega. \end{aligned} \tag{51}$$

Substituting (37) gives

$$\begin{aligned} E(Y, a) = \int_{2\pi/3}^{\pi} & [W(\omega) - aY(\omega)]^2 \\ & + [W(2\pi - \omega) - a + aY(\omega)]^2 \\ & + [W(2\omega) - a + aY(\omega)]^2 \\ & + [W(4\pi - 2\omega) - aY(\omega)]^2 \, d\omega. \end{aligned} \tag{52}$$

Taking the derivative of (52) with respect to Y and setting it equal to 0 will give the following optimal value for $Y(\omega)$:

$$Y(\omega) = \frac{1}{2} - \frac{1}{4a}[W(2\pi - \omega) - W(\omega) + W(2\omega) - W(4\pi - 2\omega)]. \tag{53}$$

Taking the derivative of (52) with respect to a and setting it equal to 0 will give the following optimal value for the scaling constant, a

$$a = \tfrac{1}{2} \int_{2\pi/3}^{4\pi/3} W(\omega) \, d\omega + \tfrac{1}{4} \int_{4\pi/3}^{8\pi/3} W(\omega) \, d\omega. \tag{54}$$

This result says that for the matched wavelet spectrum to be optimal, the desired signal spectrum must be scaled by a constant proportional to a weighted sum of its energy in the passband.

Once a is found, $Y(\omega)$ can be found for all $\omega \in [2\pi/3, 8\pi/3]$ using the solution given in (53) for $\omega \in [2\pi/3, \pi]$ and the conditions in (37),

$$\begin{aligned} Y(2\pi - \omega) &= 1 - Y(\omega) \\ Y(2\omega) &= 1 - Y(\omega) \\ Y(4\pi - 2\omega) &= Y(\omega). \end{aligned} \tag{55}$$

VI. Discrete Matching Algorithm

In this section, a practical, fast, closed form discrete solution for both the wavelet spectrum *and* its group delay is constructed which yields wavelet approximations that are very closely matched to the signal of interest. The results are approximations in so far that they guarantee the orthonormal wavelet conditions are satisfied at the sample locations only. However, the algorithm produces wavelet samples that approach continuous wavelets as the sample spacing tends to 0 at the cost of computational time.

A. Matching Discrete Spectrum Amplitude

The expression in (21) needs to be converted to a discrete implementation. However, the infinite summation makes it impossible to achieve exact results. Therefore, we have developed a recursive equation that implements the condition of (21) with exact results.

Theorem 4 (Finding $|\Phi(k)|$ from $|\Psi(k)|$): In an orthonormal MRA, let $\Phi(k\Delta\omega)$ and $\Psi(k\Delta\omega)$ be the sampled scaling function and wavelet spectra, respectively, with sample spacing $\Delta\omega = \pi/2^\ell$. Any sample of $|\Phi|$ at $\omega = k\pi/2^\ell$ can be expressed by the following recursive equation:

$$\left|\Phi\left(\frac{\pi k}{2^\ell}\right)\right|^2 = \left|\Phi\left(\frac{\pi k}{2^{\ell-1}}\right)\right|^2 + \left|\Psi\left(\frac{\pi k}{2^{\ell-1}}\right)\right|^2 \qquad \text{for } k \neq 0 \tag{56}$$

which leads to the following closed form solution

$$\left|\Phi\left(\frac{\pi k}{2^\ell}\right)\right|^2 = \sum_{p=0}^{\ell} \left|\Psi\left(\frac{2\pi k}{2^p}\right)\right|^2 \qquad \text{for } k \neq 0. \tag{57}$$

Furthermore, (56) implies $|\Psi(4\pi k)| = 0$ for all $k \in \mathbb{Z}$.

Proof: Substituting $\omega = \pi n$ in (20) gives

$$|\Phi(\pi n)|^2 = |\Phi(2\pi n)|^2 + |\Psi(2\pi n)|^2. \tag{58}$$

However, since $\sum |\Phi(\omega + 2\pi n)|^2 = 1$ for orthonormal MRA's, and $\Phi(0) = 1$, and $\Psi(0) = 0$ then, $|\Phi(2\pi n)| = 1$ for $n = 0$ and $|\Phi(2\pi n)| = 0$ for $n \neq 0$. Now (58) can be rewritten as

$$|\Phi(\pi n)| = \begin{cases} 1, & \text{for } n = 0 \\ |\Psi(2\pi n)|, & \text{for } n \neq 0 \end{cases} \tag{59}$$

So, at integer multiples of π, $|\Phi|$ can be computed directly from values of $|\Psi|$. Furthermore, (59) and the above derivation imply $|\Psi(4\pi n)| = 0$. Substituting for $\omega = \pi n/2$ in (20) gives

$$\left|\Phi\left(\frac{\pi n}{2}\right)\right|^2 = |\Phi(\pi n)|^2 + |\Psi(\pi n)|^2, \qquad \text{for } n \neq 0 \tag{60}$$

At integer multiples of $\pi/2$, $|\Phi|$ can be computed from values of $|\Psi|$ and the previously calculated values of $|\Phi|$. Repeated substitutions leads to the closed form solution in (57). If $|\Psi(k\Delta\omega_\Psi)|$ has a sample spacing of $\Delta\omega_\Psi = 2\pi/2^\ell$, then by (57), $|\Phi(k\Delta\omega_\phi)|$ has a minimum sample spacing of $\Delta\omega_\Psi = 2\pi/2^{\ell+1}$ and ℓ can take on values of $\ell = \{0, 1, \cdots, \ell\}$ □

The next step, similar to the conditions in (37) for the continuous case, is to develop the necessary and sufficient condition on $|\Psi(k\Delta\omega)|$ that guarantees orthnormality. Let

$$Y(k) = |\Psi(k\Delta\omega)|^2 \qquad k \in \mathbb{Z} \tag{61}$$

where $\Delta\omega = 2\pi/2^\ell$. The necessary and sufficient condition on Y to guarantee that $|\Phi(k)|$, found in Theorem 4, generates an orthonormal MRA can be found by substituting (57) into the Poisson summation (9). The result is given as follows:

$$\sum_{p=0}^{\ell} \sum_{m=-\infty}^{\infty} Y\left(\frac{2^{\ell}}{2^{p}}(k+2^{\ell+1}m)\right) = 1 \tag{62}$$

where

$$2^{\ell-1}/3 < \left|\frac{2^{\ell}}{2^{p}}(k+2^{\ell+1}m)\right| < 2^{\ell+2}/3 \tag{63}$$

and $\Delta\omega = \pi/2^{\ell}$ is the sample spacing of $\Phi(k\Delta\omega)$ and $\Psi(k\Delta\omega)$. To determine the specific set of constraint equations, first expand the summation over p. As an example, let $\ell = 4$ so that $\Delta\omega = 2\pi/16$. Then (62) becomes

$$\sum_{m=-\infty}^{\infty} Y[16(k+32m)] + Y[8(k+32m)] + Y[4(k+32m)] + Y[2(k+32m)] + Y(k+32m) = 1. \tag{64}$$

Because Y is nonzero on the intervals $6 \leq |k| \leq 21$, for our example, then only one or two values of m in (64) yield nonzero terms. Furthermore, since we are concerned with designing real wavelets only, the magnitude of the wavelet spectrum is even, $|\Psi(k\Delta\omega)| = |\Psi(-k\Delta\omega)|$, and we need only match the spectra for positive frequency indices, k, in the passband. For instance, when $k = 1$, then the only nonzero terms in (64) occur when $m = 0$ and they are $Y(16)$ and $Y(8)$. Therefore, $Y(8) + Y(16) = 1$ is a constraint equation for our example. When $k = 11$, we get nonzero values when $m = -1$ and $m = 0$, so that $Y(11) + Y(-21) = 1$, and since Y is even, the constraint equation can be rewritten as $Y(11) + Y(21) = 1$. A complete set of linear constraint equations can be constructed using this technique.

The condition given in (62) generates a set of L linear equality constraints in $Y(k)$ of the form

$$\sum_{i=1}^{L} \alpha_{ik} Y(k) = 1 \qquad \text{for } k = \{\lceil 2^{\ell}/3 \rceil, \cdots, \lfloor 2^{\ell+2}/3 \rfloor\} \tag{65}$$

where $\alpha_{ik} \in \{0, 1, 2\}$. Condition (65) can be expressed in vector notation as

$$\mathbf{AY} = \mathbf{1} \tag{66}$$

where $\mathbf{A}$ is a $L \times 2^{\ell}$ matrix given by

$$\mathbf{A} = \{\alpha_{ij} \in \{0, 1, 2\};\ i = 1, \cdots, L;\ j = 1, \cdots, 2^{\ell}\} \tag{67}$$

and $\mathbf{1}$ is a $L \times 1$ vector given by $\mathbf{1} = \{1\ 1\ \cdots\ 1\}$. In the following theorem, an expression for the optimal wavelet amplitude spectrum samples is given.

Theorem 5 (Matched Wavelet Amplitude): Let $\mathbf{W}$ and $\mathbf{Y}$ be vectors containing the samples of $|F(k\Delta\omega)|^2$ and $|\Psi(k\Delta\omega)|^2$, respectively, in the passband

$$\mathbf{W} = \{|F(k\Delta\omega)|^2;\ k = \lceil 2^{\ell}/3 \rceil, \cdots, \lfloor 2^{\ell+2}/3 \rfloor\} \tag{68}$$

$$\mathbf{Y} = \{|\Psi(k\Delta\omega)|^2;\ k = \lceil 2^{\ell}/3 \rceil, \cdots, \lfloor 2^{\ell+2}/3 \rfloor\} \tag{69}$$

where $F(\omega)$ is the spectrum of the signal for which we desire a matched wavelet and $\Psi(\omega)$ is the matched wavelet spectrum. If the error to be minimized is given by

$$E = \frac{(\mathbf{W} - a\mathbf{Y})^T(\mathbf{W} - a\mathbf{Y})}{\mathbf{W}^T\mathbf{W}} \tag{70}$$

then the optimal wavelet power spectrum is given by the following expression:

$$\mathbf{Y} = \frac{1}{a}\mathbf{W} + \mathbf{A}^T(\mathbf{AA}^T)^{-1}\left(\mathbf{1} - \frac{1}{a}\mathbf{AW}\right) \tag{71}$$

where

$$a = \frac{\mathbf{1}^T(\mathbf{AA}^T)^{-1}\mathbf{AW}}{\mathbf{1}^T(\mathbf{AA}^T)^{-1}\mathbf{1}} \tag{72}$$

where $(\mathbf{AA}^T)$ is full rank. The match error is given by

$$E = \frac{\left(\mathbf{1} - \frac{1}{a}\mathbf{AW}\right)^T(\mathbf{AA}^T)^{-1}\left(\mathbf{1} - \frac{1}{a}\mathbf{AW}\right)}{\frac{1}{a^2}\mathbf{W}^T\mathbf{W}}. \tag{73}$$

The resultant wavelet is orthonormal, and the scaling function it generates by way of (57) generates an orthonormal MRA.

Proof: This constrained optimization problem can be solved using Lagrangian multipliers where (73) is the error function to be minimized and (66) is the constraint equation. Solving the Lagrangian, given as

$$L = \frac{(\mathbf{W} - a\mathbf{Y})^T(\mathbf{W} - a\mathbf{Y})}{\mathbf{W}^T\mathbf{W}} + \lambda^T(\mathbf{AY} - \mathbf{1}) \tag{74}$$

yields the solutions given in (71)–(73). The matrix $(\mathbf{AA}^T)$ can be shown to be full rank since $\mathbf{A}$ is upper triangular, and therefore its rows are linearly independent [27]. The full proof of Theorem 5 can be found in [14]. □

Notice that the error, E, in (73) has the form of a Mahalonobis distance, where $(\mathbf{W}^T\mathbf{W}/a^2)\mathbf{AA}^T$ acts like a covariance matrix [26]. This implies that the solution is "closest" to the desired signal spectrum where the distance measure is given in (70).

We have solved the first half of the problem posed by (19), that of finding the optimal wavelet spectrum amplitude with respect to the input spectrum. The next section develops the algorithm for matching the discrete wavelet group delay (negative derivative of the phase) to that of the desired signal.

B. Matching Discrete Spectrum Phase

Matching the group delay of a desired signal to the group delay of a wavelet given in (45) cannot be done in the same manner as the amplitude matching since there are additional periodicity constraints on $\lambda(\omega)$. Furthermore, we still have the problem of finding the phase of Φ from the phase of Ψ. To solve both problems, we model one period of $\lambda(\omega)$, denoted as $\lambda_T(\omega)$, as a polynomial of order R. Because $\lambda(\omega)$ is an even function, the polynomial has only even exponents

$$\lambda_T(\omega) = \sum_{r=0}^{R/2} c_r \omega^{2r} \Pi\left(\frac{\omega}{2\pi}\right) \tag{75}$$

$$\begin{bmatrix}
1&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&2&0&0&0&0&0
\end{bmatrix}$$

Fig. 1. Constraint matrix-A.

where c_r are the coefficients of the polynomial and $\Pi(\omega)$ is the "rect" function defined as

$$\Pi(\omega) = \begin{cases} 1, & -\frac{1}{2} \le \omega < \frac{1}{2} \\ 0, & \text{elsewhere.} \end{cases} \tag{76}$$

Now we construct $\lambda(\omega)$ by replicating $\lambda_T(\omega)$ every 2π,

$$\begin{aligned} \lambda(\omega) &= \sum_{k=-\infty}^{\infty} \lambda_T(\omega - 2\pi k) \\ &= \sum_{k=-\infty}^{\infty} \sum_{r=0}^{R/2} c_r(\omega - 2\pi k)^{2r} \Pi\left(\frac{\omega - 2\pi k}{2\pi}\right). \end{aligned} \tag{77}$$

Like the amplitude matching algorithm, we will develop the phase matching algorithm for discrete samples of the spectrum. Let $\Delta\omega = 2\pi/T$, and N be the number of samples in $F(n\Delta\omega)$. Equation (77) can be rewritten in discrete form as

$$\lambda(n) = \sum_{r=0}^{R/2} c_r \sum_{k=-P/2}^{P/2-1} (n - kT)^{2r} \Pi\left(\frac{n - kT}{T}\right) \tag{78}$$

where $P = N/T$ is the number of periods over N samples and $-N/2 \le n < N/2$. The discrete form for $\lambda(n)$ can now be written in vector notation

$$\lambda = \mathbf{B}\mathbf{c} \tag{79}$$

where λ is a $N \times 1$ vector, $\mathbf{B}$ is a $N \times (R/2 + 1)$ matrix and $\mathbf{c}$ is a $(R/2 + 1) \times 1$ vector. The elements of $\mathbf{B}$ are given as

$$b_{n,r} = \sum_{k=-P/2}^{P/2-1} (n - kT)^{2r} \Pi\left(\frac{n - kT}{T}\right). \tag{80}$$

Substituting (79) into (45) gives a matrix equation for $\mathbf{\Gamma}_\Psi$

$$\mathbf{\Gamma}_\Psi = \mathbf{D}_\Psi \mathbf{c} \tag{81}$$

where

$$\mathbf{D}_\Psi = -\tfrac{1}{2}\mathbf{B}_{(q+T/2)} + \sum_{m=2}^{\infty} 2^{-m}\mathbf{B}_{(q/2^m)} \tag{82}$$

and the elements of $\mathbf{B}_{(q+T/2)}$ and $\mathbf{B}_{(q/2^m)}$ are given in (80) where $n = (q + T)/2$ and $n = q/2^m$, respectively.

Now all that is left is to derive the expression for $\mathbf{\Gamma}_\Psi$ that is closest to the desired signal's group delay, $\mathbf{\Gamma}_F$, in a least squares sense and yet has the structure of (81) and (82). Let γ be the unweighted error that we wish to minimize

$$\gamma = \sum_{n=-N/2}^{N/2-1} (\Gamma_F(n) - \Gamma_\Psi(n))^2. \tag{83}$$

Since the wavelet phase need only match that of the desired signal in the passband, we need to weight the error function by a normalized weighting function. Let $\Omega(n) = Y(n)/\sum Y(n)$, where $Y(n)$ are the elements of $\mathbf{Y}$ derived in Theorem 5. The weighted error function becomes

$$\gamma_\Omega = \sum_{n=-N/2}^{N/2-1} [\Omega(n)(\Gamma_F(n) - \Gamma_\Psi(n))]^2. \tag{84}$$

Rewriting (84) in vector notation gives

$$\gamma = (\overline{\mathbf{\Gamma}}_F - \overline{\mathbf{D}}_\Psi\mathbf{c})^T(\overline{\mathbf{\Gamma}}_F - \overline{\mathbf{D}}_\Psi\mathbf{c}) \tag{85}$$

where the elements of $\overline{\mathbf{\Gamma}}_F$ are the nonzero values of $\{\Omega(n)\Gamma_F(n)\}$ and the elements of $\overline{\mathbf{D}}_\Psi$ are the corresponding nonzero values of $\{\Omega(n)d_{n,r}\}$. The vector, $\hat{\mathbf{c}}$, which minimizes γ is found by setting $\nabla_c\gamma = 0$

$$\hat{\mathbf{c}} = (\overline{\mathbf{D}}_\Psi^T\overline{\mathbf{D}}_\Psi)^{-1}\overline{\mathbf{D}}_\Psi^T\overline{\mathbf{\Gamma}}_F. \tag{86}$$

The symmetric matrix, $\overline{\mathbf{D}}_\Psi^T\overline{\mathbf{D}}_\Psi$ is full rank if and only if its columns are linearly independent [27]. Because the columns of $\overline{\mathbf{D}}_\Psi$ are based on a geometric series of the column index, r, they must be linearly independent. Therefore, $\overline{\mathbf{D}}_\Psi^T\overline{\mathbf{D}}_\Psi$ is full rank and its inverse exists.

It follows that the group delay of the wavelet can be found by substituting (86) into (81)

$$\mathbf{\Gamma}_\Psi = \mathbf{D}_\Psi\hat{\mathbf{c}} \tag{87}$$

Since we have the best estimate of $\mathbf{c}$, we can find λ and calculate $\mathbf{\Lambda}_\Psi$ and $\mathbf{\Lambda}_\Phi$ directly,

$$\lambda = \mathbf{B}\hat{\mathbf{c}} \tag{88}$$

$$\begin{aligned} \mathbf{\Lambda}_\Psi &= (\mathbf{D}_\Psi\hat{\mathbf{c}} - \overline{\mathbf{D}_\Psi\hat{\mathbf{c}}}) - \frac{\Delta\omega}{2} \\ \mathbf{\Lambda}_\Phi &= \mathbf{D}_\Phi\hat{\mathbf{c}} - \overline{\mathbf{D}_\Phi\hat{\mathbf{c}}} \end{aligned} \tag{89}$$

where $\overline{\mathbf{D}_\Psi\hat{\mathbf{c}}}$ and $\overline{\mathbf{D}_\Phi\hat{\mathbf{c}}}$ are the means of $\mathbf{D}_\Psi\hat{\mathbf{c}}$ and $\mathbf{D}_\Phi\hat{\mathbf{c}}$, respectively, and by (43) and (79)

$$\mathbf{D}_\Phi = \sum_{m=1}^{\infty} 2^{-m}\mathbf{B}_{(q/2^m)}. \tag{90}$$

We subtract the means in (89) so that $\mathbf{\Lambda}_\Psi$ and $\mathbf{\Lambda}_\Phi$ have the properties of Theorem 3. Both $\mathbf{\Lambda}_\Psi$ and $\mathbf{\Lambda}_\Phi$ can be summed to obtain the discrete phases of Φ and Ψ that when combined with the magnitudes from Theorem 5 give the full estimate of $\Phi(n\Delta\omega)$ and $\Psi(n\Delta\omega)$ which satisfy all conditions for an orthonormal MRA. The QMF filter pair impulse responses, h and g, corresponding to the matched wavelet and its scaling function can be found using (12) and the inverse Fourier transform. A flow chart of the complete algorithm has been provided in more detail in previous publications [14].

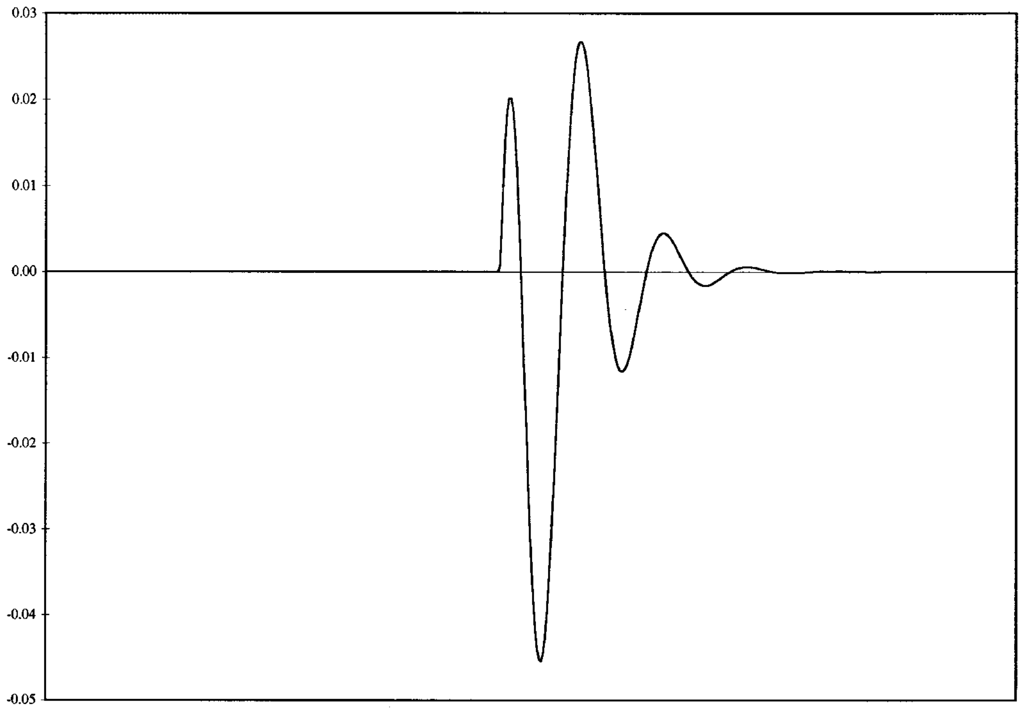

Fig. 2. Transient signal.

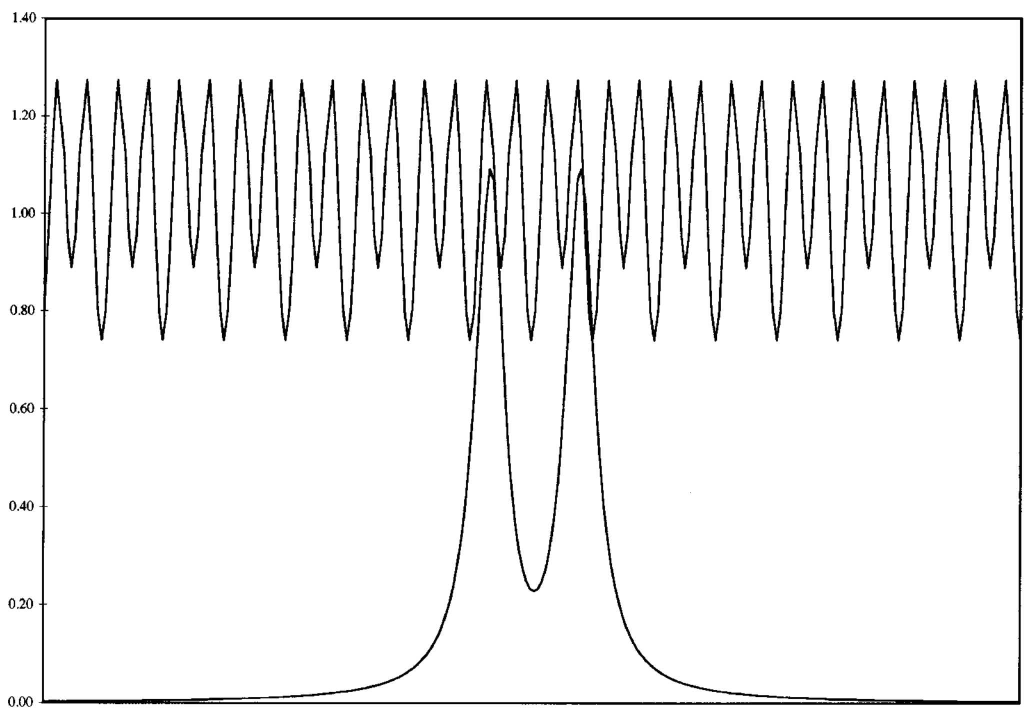

Fig. 3. Desired signal spectrum and Poisson sum—transient.

VII. EXAMPLES

In this section, we will demonstrate the performance of both the magnitude and phase matching algorithms with an example. The amplitude matching algorithm was demonstrated on symmetric signals (zero phase), including Meyer's wavelet and a nonorthonormal, truncated Gaussian, in previous conference papers [28], [29]. Here we will demonstrate both the magnitude and group delay matching algorithms' performance on an asymmetric signal. In the following example, we set $N = 512$, and $\Delta\omega = 2\pi/16$ so that $\ell = 4$. In each of the figures shown, the input signal is a dotted line and the matched signal is a solid line. With $\ell = 4$, the nonzero frequency indices in (65) are $k = \{6, 7, \cdots, 21\}$. The equality constraints in (62) and (63)

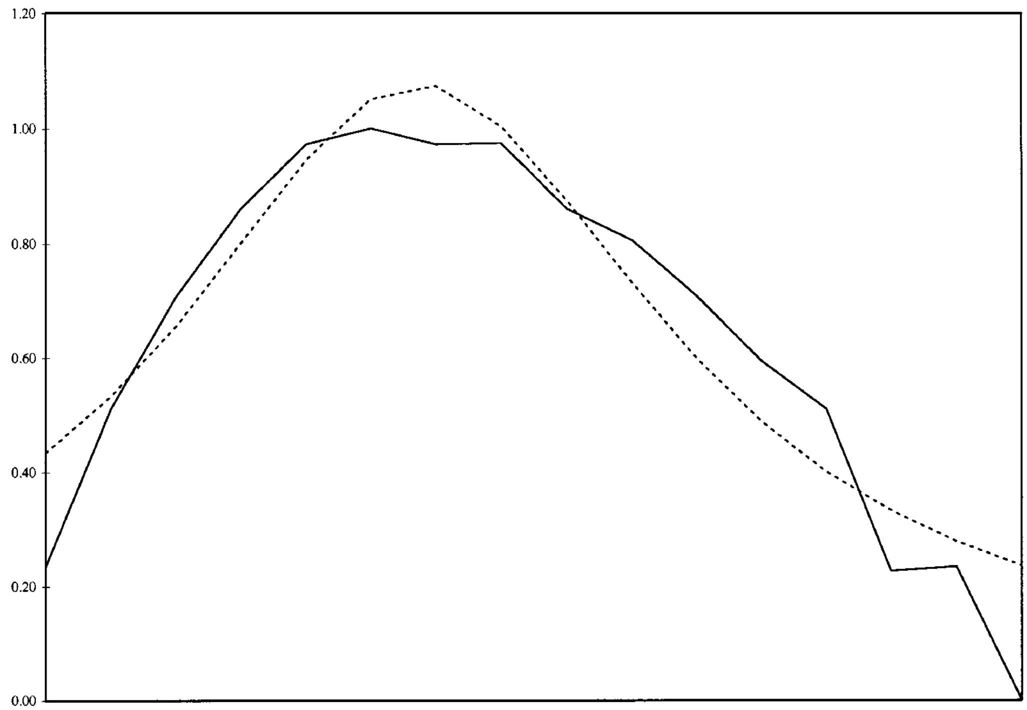

Fig. 4. Amplitude match in the passband—transient.

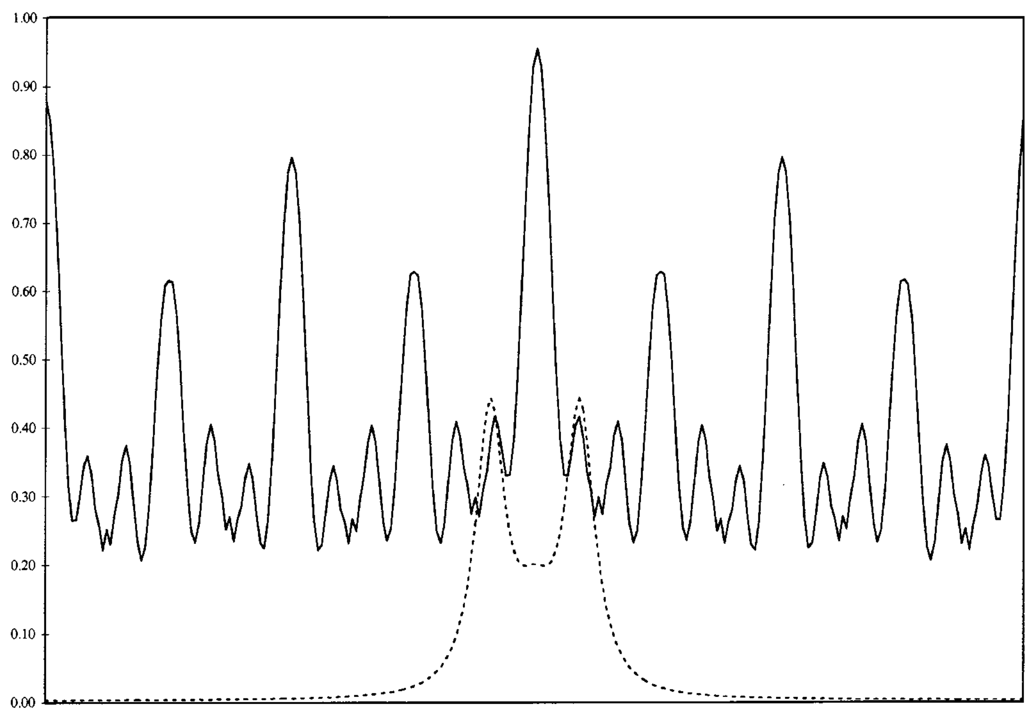

Fig. 5. Matched wavelet group delay versus desired—transient.

generate $L = 11$ equations and $2^\ell = 16$ unknowns represented by the $\mathbf{A}$ matrix in Fig. 1. We will use this matrix for matching a wavelet to the transient signal. For our example, we use a transient sinusoid given by the following equation:

$$f_T(x) = xe^{\alpha x} \cos(2\pi f_0 x)u(x) \tag{91}$$

where $u(x)$ is the unit-step function. The transient signal in this example was constructed by setting $\alpha = 2.0$ and $f_0 = 0.8$, and dilating it such that its spectrum, $F_T(\omega)$, had maximum energy in the passband, $2\pi/3 \leq |\omega| \leq 8\pi/3$. Fig. 2 shows the transient signal. The transient signal amplitude spectrum and it's associated Poisson summation show in Fig. 3 that the signal is clearly nonorthonormal. Fig. 4 shows the result of matching the wavelet in the positive passband, where $a = 1.0348$. The group delays of the desired signal, $\mathbf{\Lambda}_F$, and the matched wavelet, $\mathbf{\Lambda}_\Psi$, are shown in Fig. 5. Since $f_T(x)$ is not a wavelet, we wouldn't

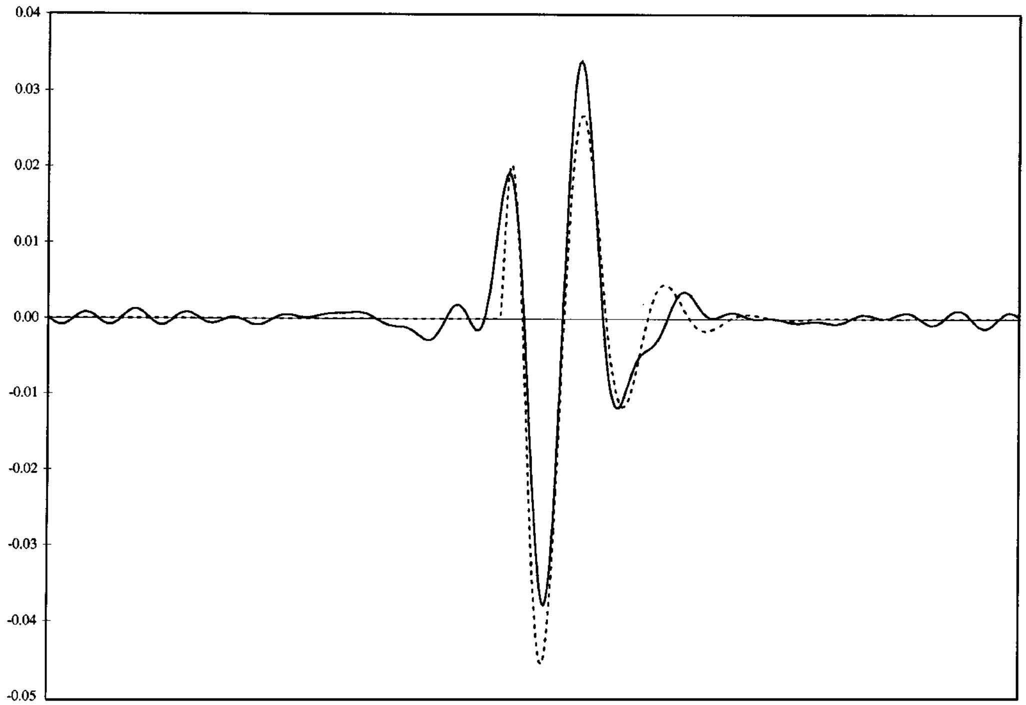

Fig. 6. Matched wavelet versus desired signal—transient.

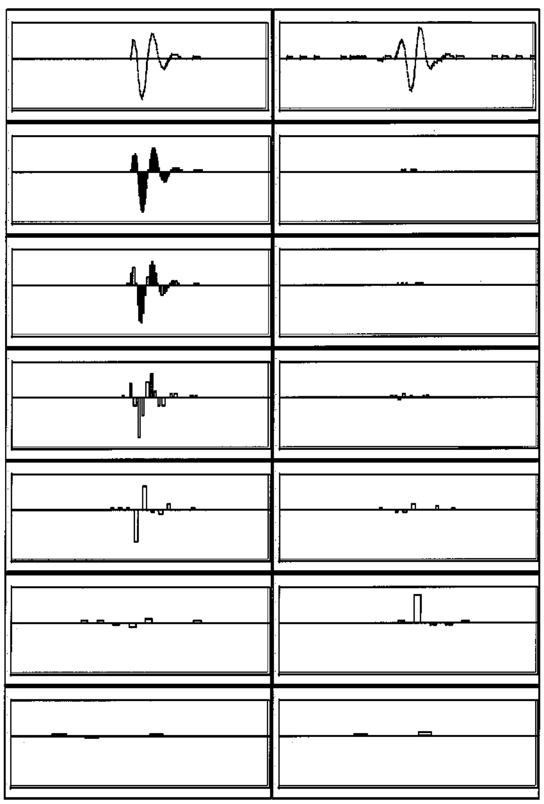

Fig. 7. Discrete wavelet decomposition of transient signal.

expect its phase to have the required structure. However, notice that the matched wavelet group delay does have the required structure *and* matches the desired group delay very well in the passband. The matched wavelet is shown in Fig. 6. The inner product of f_T with its matched wavelet, ψ gives $\{\cdots\ -0.0062\ -0.0015\ 0.9834\ -0.0412\ -0.0935\ \cdots\}$. Even though f_T is not bandlimited, its correlation with the matched wavelet still produces a value very near 1.0, with very little spread in translation. Therefore, constraining the matched wavelets to be bandlimited is not a significant impediment, but in fact should provide very good segmentation across scales. Fig. 7 show the wavelet decomposition of the transient signal using the matched wavelet. Matching clearly results in a prominent peak at the appropriate time and scale location.

VIII. Summary

In this paper, we have further developed the bandlimited orthonormal wavelet for applications requiring a matched filter approach. We have shown that for bandlimited orthonormal wavelets, Meyer's spectrum amplitude construction is not only sufficient, but necessary. We have developed closed form expressions in the continuous domain for directly matching wavelet spectrum amplitudes to that of a desired signal. We have developed a fast numerical algorithm for finding matched wavelet amplitude spectra and group delays using sampled data that yield good matched wavelets. In the future, we will begin to apply this algorithm to image processing applications and communications applications where signal detection and signal enhancement are required.

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